1. (a) `xat` plotted w.r.t. `t`

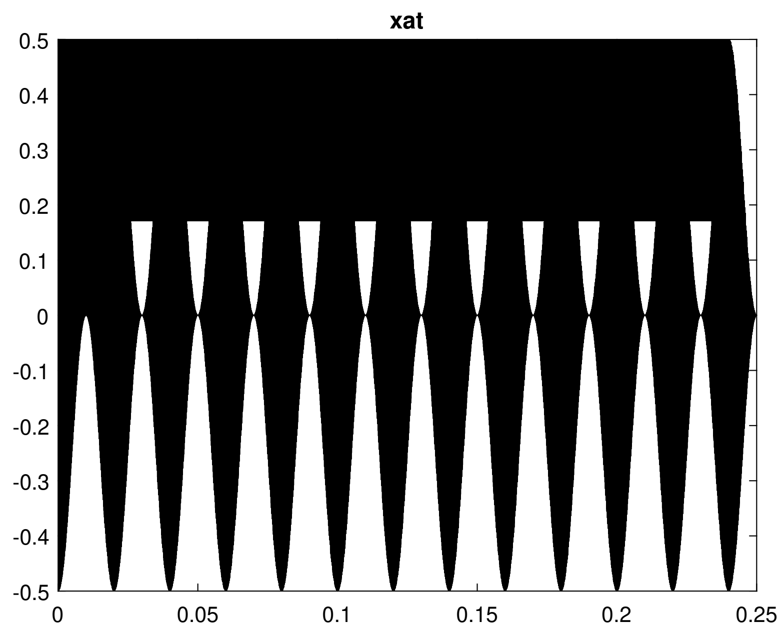


Figure 1: Plot of bandpass signal `xat`

(b) Calculating appropriate `Fs`

```
%% part b
% Fs must ≥ 2*bandwidth, bandwidth = 1950
Fc = 65000;
k = floor(Fc / B - 0.5);
Fs = 4 * Fc / (2 * k + 1);

% For Fc = 65kHz, B = 1.95kHz: Fs = 4kHz
```

(c) Calculating `xanT`

```
%% part c, slides approach
% calculate time variables
ΔT = 1e-6;
tfinal = 0.25;
t = 0:ΔT:tfinal;

% calculate sampling interval and how many samples
Ts = 1 / Fs;
nfinal = tfinal / Ts;
n = 0:nfinal;

% sample analog signal at rate of sampling interval, plot
xanT = xat(1:round(Ts / ΔT):end);
stem(t(1:round(Ts / ΔT):end), xanT);
```

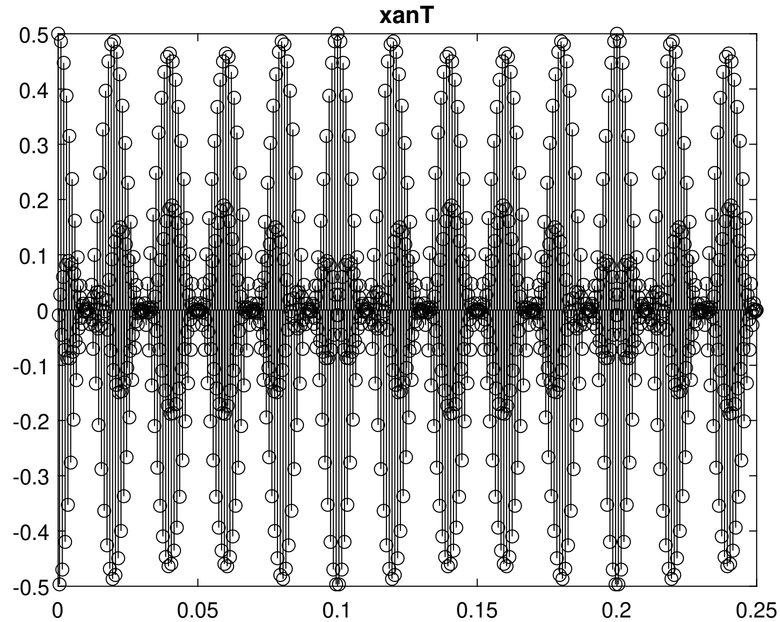


Figure 2: Stem of sampled `xanT`

NOTE: *This method, which is outlined in the slides, is inflexible when it comes to changing* `Fc`*,* `B`*, and* `deltaT`*; there is inconsistency with the sampling and matching of indicies when those variables fluctuate. Below I have a more flexible approach, which works all of the time, and yields the same absolute error at the end as the textbook approach:*

```
%% part c, my approach
% calculate time variables
ΔT = 1e-6;
tfinal = 0.25;
t = 0:ΔT:tfinal;

% calculate sampling interval and how many samples
Ts = 1 / Fs;
nfinal = tfinal / Ts;
n = 0:nfinal;

% preallocate memory for indicies of analog signal
nT_i = zeros(size(n));

% loop through and find indicies where n*Ts = t
for i = n
    nT_i(i + 1) = find(round(i * Ts, 6) == t);
end

% sample at indicies where n*Ts = t, plot
xanT = xat(nT_i);
nT = t(nT_i);
stem(nT, xanT);
```

(d) Calculating `xIn` and `xQn`

```
%% part d
% multiply xanT by cos(0.5 pi n) for all even indicies to get xIn
xIn = xanT(1:2:end) .* cos(0.5 * pi * n(1:2:end));
% multiply xanT by sin(0.5 pi n) for all odd indicies to get -xQn
xQn = -xanT(2:2:end) .* sin(0.5 * pi * n(2:2:end));
```

(e) Plot of `xIn` and `xQn`

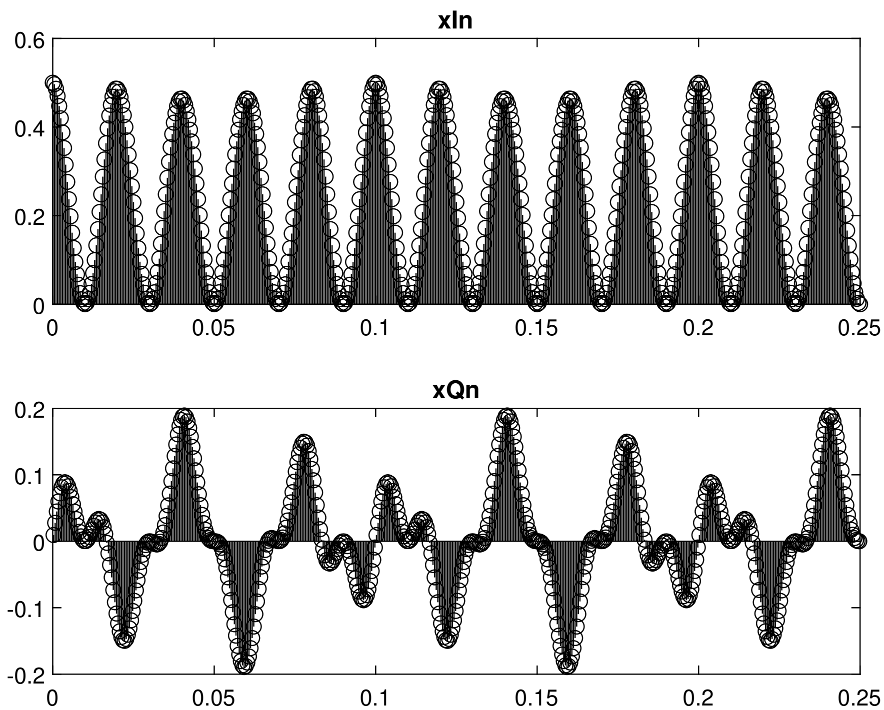


Figure 3: Stem of `xIn` above `xQn`

(f) Calculating absolute error between in-phase component and the variable `xIn_true`, as well as the quad-phase component with the variable `xQn_true`

```
%% part f
% absolute error: xIn
error_I = abs(xIn - xIn_true);
% absolute error: xQn
error_Q = abs(xQn - xQn_true);
```

(g) Plot of absolute errors calculated in part (f).

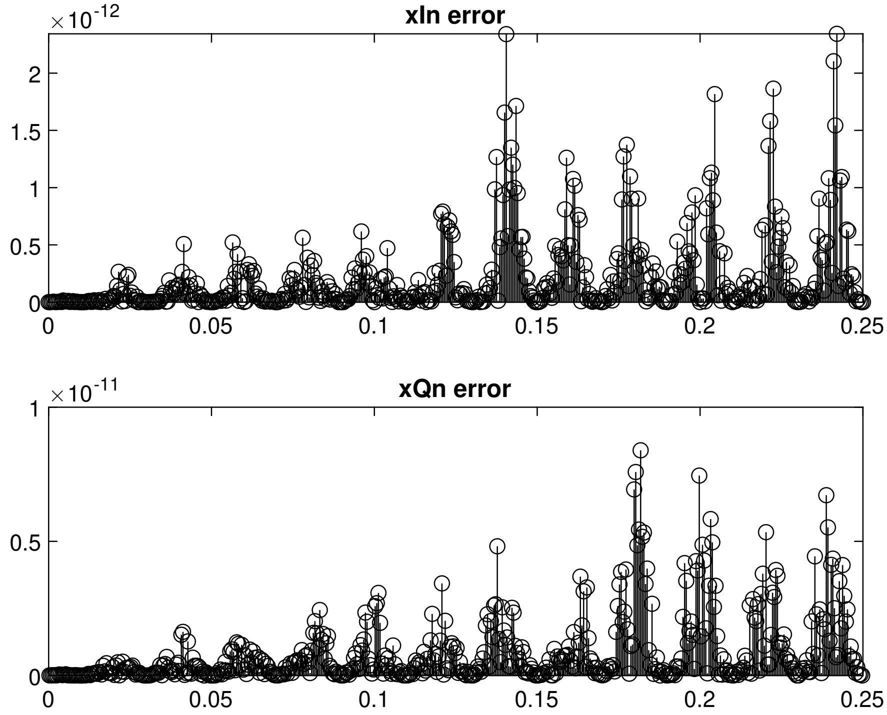


Figure 4: Absolute error of `xIn` and `xQn`

2. Given that $x(n)$ is defined as

$$x(n) \equiv x_a(nT_s) = \int_{(m-1)B}^{mB} X_a(F)e^{j2\pi Ft}dF + \int_{-mB}^{-(m-1)B} X_a(F)e^{j2\pi Ft}dF$$

And we know that the spectrum of the analog signal is represented by

$$X_a(F) = \frac{1}{F_s}X(F), (m-1)B < |F| < mB$$

We can rewrite the identity as

$$x_a(nT_s) = x_a(t) = \int_{(m-1)B}^{mB} \left[\frac{1}{F_s}X(F)\right] e^{j2\pi Ft}dF + \int_{-mB}^{-(m-1)B} \left[\frac{1}{F_s}X(F)\right] e^{j2\pi Ft}dF$$

$X(F)$ is the frequency domain representation of discrete time signal $x(n)$, meaning that the following discrete Fourier transform equates the two

$$X(F) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\frac{2\pi F_n}{F_s}}$$

Plug that in to what we have above to get

$$\frac{1}{F_s}\int_{(m-1)B}^{mB} \left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\frac{2\pi F_n}{F_s}}\right] e^{j2\pi Ft}dF + \frac{1}{F_s}\int_{-mB}^{-(m-1)B} \left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\frac{2\pi F_n}{F_s}}\right] e^{j2\pi Ft}dF$$

$$\frac{1}{F_s}\left[\int_{(m-1)B}^{mB}\left(\sum_{n=-\infty}^{\infty}x(n)\right)e^{j2\pi F\left(t-\frac{n}{F_s}\right)}dF+\int_{-mB}^{-(m-1)B}\left(\sum_{n=-\infty}^{\infty}x(n)\right)e^{j2\pi F\left(t-\frac{n}{F_s}\right)}dF\right]$$

Substitute $\frac{1}{F_s}=T_s$ inside the integral. Pull out summations, since independent of F

$$\frac{1}{F_s}\sum_{n=-\infty}^{\infty}x(n)\left[\int_{(m-1)B}^{mB}e^{j2\pi F(t-nT_s)}dF+\int_{-mB}^{-(m-1)B}e^{j2\pi F(t-nT_s)}dF\right]$$

Evaluate the integral

$$\frac{1}{F_s}\sum_{n=-\infty}^{\infty}x(n)\left[\frac{e^{j2\pi F(t-nT_s)}}{j2\pi(t-nT_s)}\bigg|_{(m-1)B}^{mB}+\frac{e^{j2\pi F(t-nT_s)}}{j2\pi(t-nT_s)}\bigg|_{-mB}^{-(m-1)B}\right]$$

$$\frac{1}{F_s}\sum_{n=-\infty}^{\infty}x(n)\frac{1}{\pi(t-nT_s)}\left(\frac{1}{2j}\right)\left[\left(e^{j(mB)(2\pi)(t-nT_s)}-e^{-j(mB)(2\pi)(t-nT_s)}\right)\right.$$
$$\left.+\left(e^{j(B-mB)(2\pi)(t-nT_s)}-e^{-j(B-mB)(2\pi)(t-nT_s)}\right)\right]$$

$$\frac{1}{F_s}\sum_{n=-\infty}^{\infty}x(n)\frac{1}{\pi(t-nT_s)}\left(\sin 2\pi(mB)(t-nT_s)+\sin 2\pi(B-mB)(t-nT_s)\right)$$

Substitute

$$F_s=2B, m=\frac{F_c}{B}+\frac{1}{2}$$

$$\frac{1}{2B}\sum_{n=-\infty}^{\infty}x(n)\frac{1}{\pi(t-nT_s)}\left(\sin 2\pi(F_c+\frac{B}{2})(t-nT_s)+\sin 2\pi(-F_c+\frac{B}{2})(t-nT_s)\right)$$

We use the identity

$$\sin(\alpha)+\sin(\beta)=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

to reduce the form down to

$$\sum_{n=-\infty}^{\infty}x(n)\frac{1}{2\pi B(t-nT_s)}\left(2\sin\frac{2\pi B(t-nT_s)}{2}\cos\frac{2\pi(2F_c)(t-nT_s)}{2}\right)$$

$$x_a(t)=\sum_{n=-\infty}^{\infty}x(n)\frac{\sin\pi B(t-nT_s)}{\pi B(t-nT_s)}\cos 2\pi F_c(t-nT_s)$$

3. (a) Given $x_a(t)$'s spectrum resembles a triangle function ranging from -25 to 25Hz, it will be convoluted with the $cos(2\pi 175t)$ at $\pm$175Hz and scaled down by a half. The range of $Y_a(f)$ will then be -200 to 200Hz, meaning the minimum sampling rate F_s is 400Hz.

   (b) The ideal reconstruction function $g_a(t)$'s spectrum is multiplied with $Y(f)$ in order to reconstruct the signal. Since during sampling an impulse train causes repetitions of the signal in the frequency domain, it must be low-passed to get rid of these unwanted frequencies. That's where $G_a(f)$ equals 0. The rest of $G_a(f)$ must scale $Y(f)$ back down to normal size, since during sampling it was scaled up by a factor of $F_{s,A/D}$, so this part will uniformly equal $\frac{1}{F_{s,A/D}}$.

   This means

$$G_a(f) = \begin{cases} \frac{1}{F_{s,A/D}}, & |F| < \frac{F_{s,A/D}}{2} \\ 0, & otherwise \end{cases}$$

   Where $F_{s,A/D} = 400$Hz, so that

$$G_a(f) = \frac{1}{400}\Pi\left(\frac{1}{400}\right) \Longleftrightarrow g_a(t) = \text{sinc}(400t)$$

   (c) Plot of $G_a(f)$

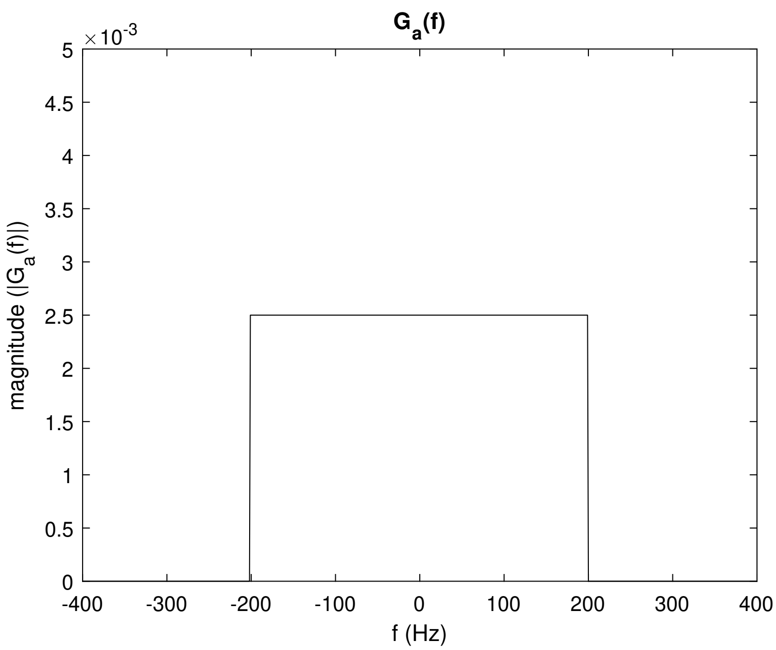


Figure 5: Spectrum of $g_a(t)$

4. Same $x_a(t)$ from 3., but with a new bandpass signal.

   (a) Plot of $Y_a(f)$

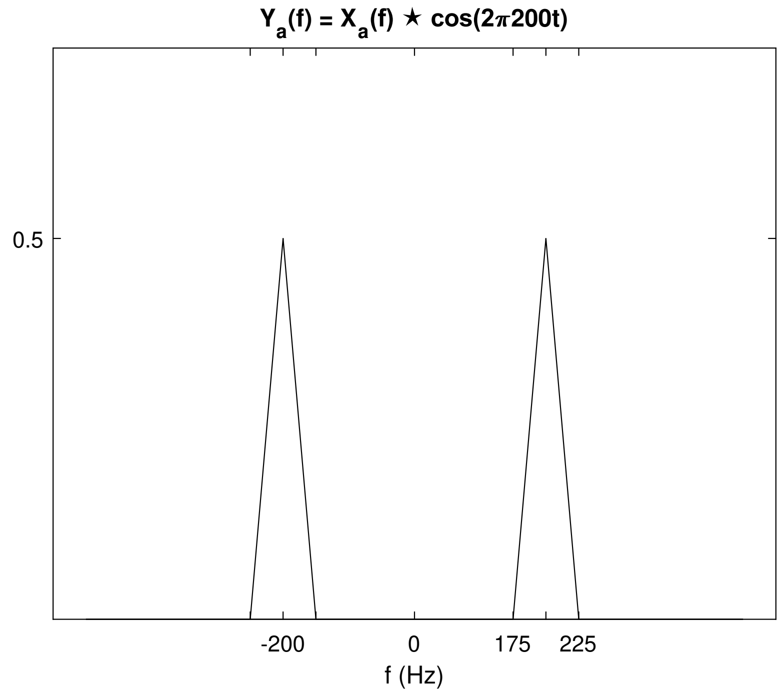


Figure 6: Spectrum of $y_a(t)$

(b) Minimum sampling rate F_s to obtain in-phase and quadrature-phase components of the bandpass signal is determined by the following formula

$$F_s = \frac{4F_c}{2k+1}$$

where $F_c = 200\text{Hz}$ (from $y_a(t)$), $k = \lfloor \frac{F_c}{B} - \frac{1}{2} \rfloor$, and $B =$ bandwidth of the bandpass signal $x_a(t) = 50\text{Hz}$. So

$$F_s = \frac{4(200)}{2(3)+1} \approx 114.29\text{Hz}$$

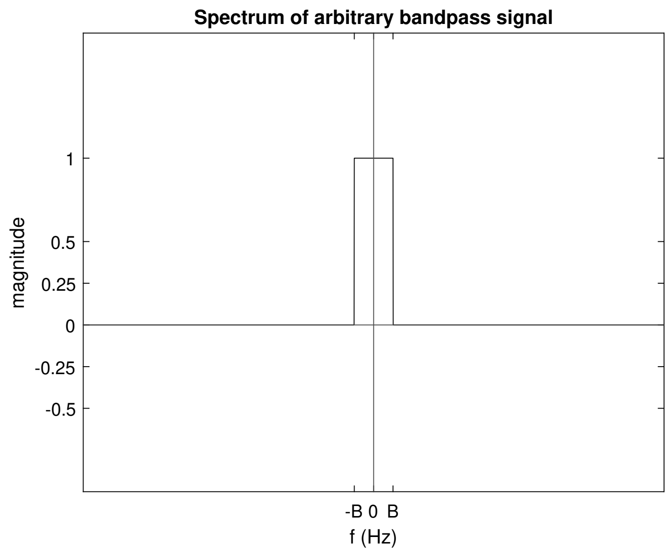
5. (a) If $x_a(t)$ undergoes quadrature demodulation, 2 sinusoids in quadrature, oscillating at the carrier frequency of $x_a(t)$, are mixed with $x_a(t)$ to get the in-phase and quadrature-phase components of $x_a(t)$, which are useful for bandpass signal reconstruction and analysis. However, after mixing, a lowpass filter must be applied to the resulting product $\hat{x}_a(t)$ to get the envelope of $\hat{x}_a(t)$, which are the in/quadrature-phase components of $x_a(t)$. It is given that the bandwidth of both $x_I(t)$ and $x_Q(t)$ do not exceed $\frac{F_c}{8}$. Therefore, the lowpass filter, $H(f)$, has the following characteristics

$$H(f) = \begin{cases} 2, & |F| < \frac{F_c}{8} \\ 0, & otherwise \end{cases}$$

where

$$Y(f) = \hat{X}_a(f)H(f)$$

The amplitude is 2 due to the fact that the bandpass signal gets modulated twice (magnitude is halved twice), but during the second modulation, the negative and positive frequencies constructively interfere to increase magnitude by 2. So if the bandpass signal starts with amplitude A, it must maintain amplitude A after demodulation. Amplitude becomes $\frac{A}{2}$ when modulated with carrier frequency, amplitude becomes $\frac{2A}{4} = \frac{A}{2}$ when modulated again for demodulation, so our lowpass filter must multiply the result by 2 to get back to A. This can be seen in the graphs below



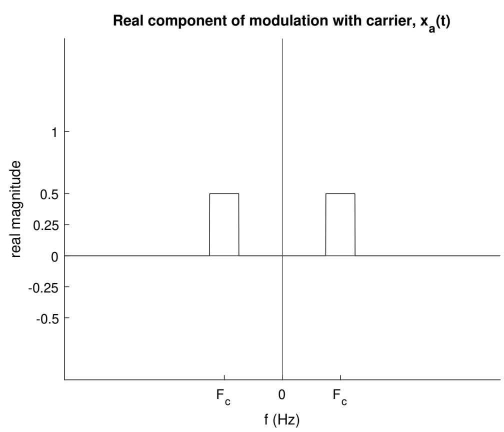


Figure 7: Real component of $x_a(t)$

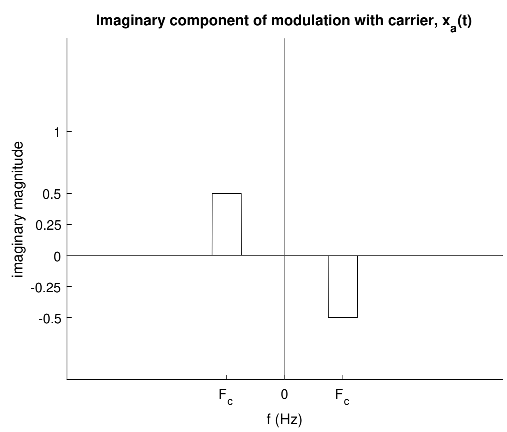


Figure 8: Complex component of $x_a(t)$

And since in this example we multiply by a cos, we get the spectrum for $\hat{x}_a(t)$:

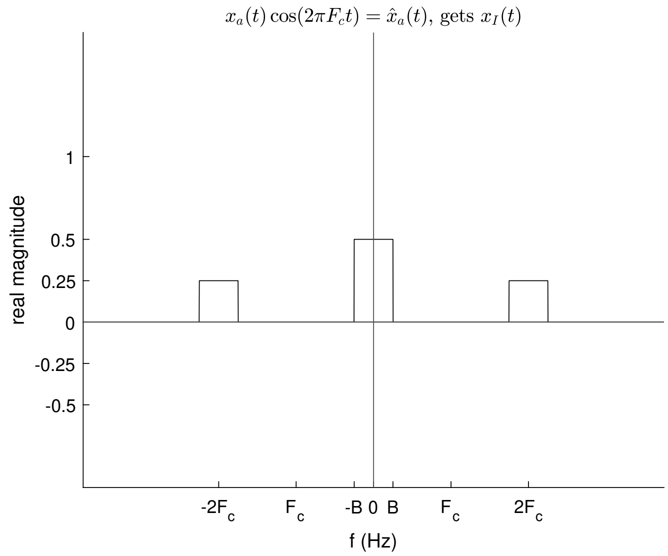


Figure 9: $B \leq \frac{F_c}{8}$

(b) Have to show the following

$$y(t) = \hat{x}_a(t) \star h(t) = -x_Q(t)$$

Since

$$x_a(t) = x_I(t)\cos(2\pi F_c t) - x_Q(t)\sin(2\pi F_c t)$$

and to find $x_Q(t)$, we demodulate using

$$\hat{x}_a(t) = x_a(t)\sin(2\pi F_c t)$$

then we can simply say

$$y(t) = [(x_I(t)\cos(2\pi F_c t) - x_Q(t)\sin(2\pi F_c t))\sin(2\pi F_c t)] \star h(t) = -x_Q(t)$$

Let's drop the convolution for now, since all $h(t)$ is just a lowpass filter, so we have

$$\hat{x}_a(t) = x_I(t)\cos(2\pi F_c t)\sin(2\pi F_c t) - x_Q(t)\sin^2(2\pi F_c t)$$

Multiplication is convolution in the frequency domain, so let's determine the trigonometric convolutions independently of $x_I(t)$ and $x_Q(t)$:

$$\cos(2\pi F_c t)\sin(2\pi F_c t) \Longleftrightarrow$$

$$\frac{1}{2}\left(\delta(f - f_c) + \delta(f + f_c)\right) \star -\frac{j}{2}\left(\delta(f - f_c) - \delta(f + f_c)\right)$$

$$-\frac{j}{4}\left[\delta(f - 2f_c) - \delta(f) + \delta(f) - \delta(f + 2f_c)\right]$$

$$-\frac{j}{4}\left[\delta(f - 2f_c) - \delta(f + 2f_c)\right]$$

And using the arbitrary bandpass signal in part (a), we can plot this to see it visually

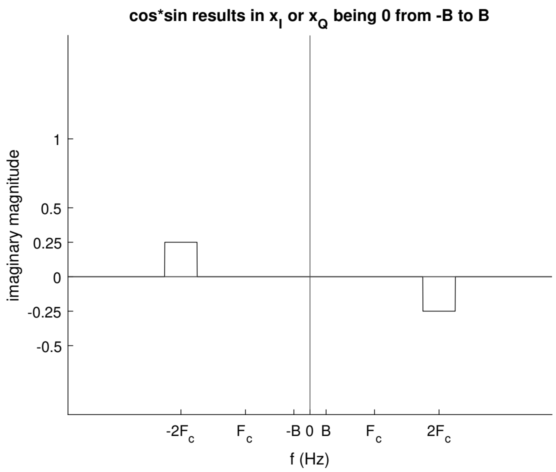


Figure 10: Component centered at 0Hz is canceled, removing the term after the lowpass filter

As for the $\sin^2$ term,

$$\sin(2\pi F_c t)\sin(2\pi F_c t) \iff$$

$$-\frac{j}{2}\left(\delta(f-f_c)-\delta(f+f_c)\right) \star -\frac{j}{2}\left(\delta(f-f_c)-\delta(f+f_c)\right)$$

$$-\frac{1}{4}\left[\delta(f-2f_c)-\delta(f)-\delta(f)+\delta(f+2f_c)\right]$$

$$-\frac{1}{4}\left[\delta(f-2f_c)-2\delta(f)+\delta(f+2f_c)\right]$$

Which we can also plot using the same technique above

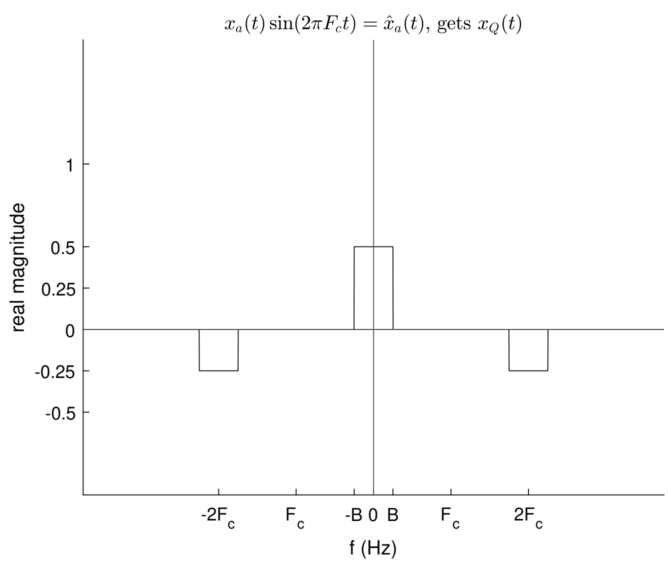


Figure 11: Component centered at 0Hz is constructive, keeping the term after the lowpass filter

So going back to our equation, we can reduce it down to

$$\hat{x}_a(t) = x_I(t)(0) - x_Q(t)(0.5), |f| \leq \frac{F_c}{8}$$

Add in the lowpass filter outlined in part (a)

$$\hat{x}_a(t) \star h(t) = 2(x_I(t)(0) - x_Q(t)(0.5))$$

$$y(t) = \hat{x}_a(t) \star h(t) = -x_Q(t)$$